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Foliation-coupling Dirac structures

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Abstract

We extend the notion of “coupling with a foliation” from Poisson to Dirac structures and get the corresponding generalization of the Vorobjev characterization of coupling Poisson structures [Yu.M. Vorobjev, Coupling tensors and Poisson geometry near a single symplectic leaf, Lie algebroids and related topics in differential geometry, Banach Center Publ., Polish Acad. Sci. (Warsaw) 54 (2001) 249–274; I. Vaisman, Coupling Poisson and Jacobi structures, Int. J. Geom. Meth. Mod. Phys. 1 (5) (2004) 607–637]. We show that any Dirac structure is coupling with the fibers of a tubular neighborhood of an embedded presymplectic leaf. We give new proofs of the results of Dufour and Wade [J.-P. Dufour, A. Wade, On the local structure of Dirac manifolds. arXiv:math.SG/0405257] on the transversal Poisson structure, and compute the Vorobjev structure of the total space of a normal bundle of the leaf. Finally, we use the coupling condition along a submanifold, instead of a foliation, in order to discuss submanifolds of a Dirac manifold which have differentiable, induced Dirac structures. In particular, we get an invariant that reminds the second fundamental form of a submanifold of a Riemannian manifold. © 2005 Elsevier B.V. All rights reserved.

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1. Introduction

In this paper, the functions, manifolds, bundles, etc. are assumed to be differentiable of class C^∞ . The Dirac structures were first defined by Courant and Weinstein [3] and

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studied in Courant’s thesis [4]. Dirac structures are important because they provide a unified view of Poisson and presymplectic structures, and generalize both. Later, Dorfman [8] extended the notion of a Dirac structure to complexes over Lie algebras. On the other hand, the bracket used by Courant was extended by Liu et al. [12] to a notion of Courant algebroid and the corresponding generalization of Dirac structures were introduced and used in [13]. An extension of the original Courant bracket, which is not a Courant algebroid bracket but includes the Jacobi structures in the scheme, was defined by Wade [21].

For the reader’s convenience, we recall the general definitions of [12,13] in a slightly different form. A *Courant algebroid* is a vector bundle $p : C \rightarrow M$ endowed with a non-degenerate, pseudo-Euclidean metric $g \in \Gamma \odot^2 C$ (Γ denotes spaces of cross-sections and \odot denotes symmetric tensor product), a morphism $\rho : C \rightarrow TM$ (the *anchor*) and a skew-symmetric bracket $[\cdot]_C : \Gamma C \times \Gamma C \rightarrow \Gamma C$ such that:

- (i) $\rho[c_1, c_2]_C = [\rho c_1, \rho c_2]_{TM}$,
- (ii) $\sum_{\text{Cycl}(1,2,3)} [[c_1, c_2]_C, c_3]_C = \frac{1}{3} \partial \{ \sum_{\text{Cycl}(1,2,3)} g([c_1, c_2]_C, c_3) \}$,
- (iii) $(\rho c)\{g(c_1, c_2)\} = g([c, c_1]_C + \partial g(c, c_1), c_2) + g(c_1, [c, c_2]_C + \partial g(c, c_2))$, where $c_a \in \Gamma C$ ($a = 1, 2, 3$) and, if $f \in C^\infty(M)$, $\partial f = (1/2)\sharp_g \circ^t \rho(df)$ (\circ^t denotes transposition and the “musical morphisms” are defined like in Riemannian geometry), equivalently,

$$g(c, \partial f) = \frac{1}{2}(\rho c)f. \tag{1.1}$$

Further basic properties of Courant algebroids may be found, for example, in [16,19].

The most important Courant algebroids are the so-called doubles of Lie bialgebroids [12,13]. We describe them by means of the notion of a *para-Hermitian structure* on a pseudo-Euclidean bundle (C, g) (e.g. [6]) that is, a bundle morphism $F : C \rightarrow C$ such that

$$F^2 = \text{Id.}, \quad g(Fc_1, Fc_2) = -g(c_1, c_2) \quad (\forall c_{1,2} \in \Gamma C). \tag{1.2}$$

A para-Hermitian vector bundle (C, g, F) decomposes as $C = C_+ \oplus C_-$, where the components are the (± 1) -eigenspaces of F , respectively, and the projections onto these components are

$$F_\pm = \frac{1}{2}(\text{Id.} \pm F). \tag{1.3}$$

Moreover, C_\pm are maximal isotropic with respect to g , the latter must be *neutral* (i.e., of signature zero), and one has isomorphisms $\flat_g : C_\pm \rightarrow C_\mp^*$ (the star denotes the dual bundle). Furthermore, the bundle also has the non-degenerate 2-form

$$\omega(c_1, c_2) = g(c_1, Fc_2), \quad (\omega(Fc_1, Fc_2) = -\omega(c_1, c_2)), \tag{1.4}$$

and the subbundles C_\pm are ω -Lagrangian. All these facts apply to para-Hermitian vector spaces, which is the case where the basis M is a point.

A Courant algebroid $(C, g, \rho, [\cdot]_C)$ will be called *para-Hermitian* if it is endowed with a para-Hermitian structure F such that the subbundles C_\pm are closed with respect to the

bracket $[\cdot, \cdot]_C$ i.e., $\forall c_1, c_2 \in \Gamma C$, one has

$$F_-[F_+c_1, F_+c_2]_C = 0, \quad F_+[F_-c_1, F_-c_2]_C = 0. \tag{1.5}$$

Taking into account the expression of F_\pm and the property $F^2 = \text{Id.}$, we see that the two conditions above are equivalent with the following single condition

$$[Fc_1, Fc_2]_C - F[Fc_1, c_2]_C - F[c_1, Fc_2]_C + [c_1, c_2]_C = 0, \tag{1.6}$$

which will be called the *integrability condition* of F , because this is the integrability condition of a para-Hermitian structure on the tangent bundle of a manifold.

Since the subbundles C_\pm are g -isotropic, the Courant algebroid axioms imply that the vector bundle structure of a para-Hermitian Courant algebroid is that of a direct sum of two dual Lie algebroids of anchors $\rho \circ F_+$, $\rho \circ F_-$. Moreover, the Lie algebroid brackets of C_\pm together with F and g determine the Courant bracket of C . Indeed, from (1.3) and (1.5), it follows that

$$\begin{aligned} F_+[c_1, c_2]_C &= [F_+c_1, F_+c_2]_C + F_+([F_+c_1, F_-c_2]_C + [F_-c_1, F_+c_2]_C), \\ F_-[c_1, c_2]_C &= [F_-c_1, F_-c_2]_C + F_-([F_+c_1, F_-c_2]_C + [F_-c_1, F_+c_2]_C). \end{aligned} \tag{1.7}$$

Then, by writing down axiom (iii) of the definition of a Courant algebroid for triples (F_+c, F_+c_1, F_-c_2) , (F_-c_2, F_-c_1, F_+c) , instead of (c, c_1, c_2) , using (1.1) and conveniently permuting (c, c_1, c_2) , we get the formulas

$$\begin{aligned} g(F_+c, [F_+c_1, F_-c_2]_C) &= g(F_-c_2, [F_+c, F_+c_1]_C) + (\rho F_+c_1)(g(F_+c, F_-c_2)) \\ &\quad - \frac{1}{2}(\rho F_+c)(g(F_+c_1, F_-c_2)), \\ g(F_-c, [F_+c_1, F_-c_2]_C) &= -g(F_+c_1, [F_-c, F_-c_2]_C) + (\rho F_-c_2)(g(F_-c, F_+c_1)) \\ &\quad + \frac{1}{2}(\rho F_-c)(g(F_+c_1, F_-c_2)). \end{aligned} \tag{1.8}$$

These formulas define the brackets $[F_+c_1, F_-c_2]_C$, which, together with (1.7), proves the previous assertion.

Accordingly, one can see that the notion of a para-Hermitian Courant algebroid is the same as that of the double of a Lie bialgebroid [12].

An *almost Dirac structure* of the para-Hermitian Courant algebroid C is a maximal g -isotropic subbundle L of C [13]. The isotropy property may be expressed by

$$g(F_+l_1, F_-l_2) + g(F_-l_1, F_+l_2) = 0 \quad \forall l_1, l_2 \in \Gamma L. \tag{1.9}$$

The algebraic properties of almost Dirac structures were discussed in [4]. Put $p_\pm = F_\pm|_L$. Then, $\ker p_\pm = C_\mp \cap L$, and we get the subbundles

$$L_\pm = \text{im } p_\pm \approx L/C_\mp \cap L.$$

It is easy to see that $\ker(\omega|_L) = (C_+ \cap L) \oplus (C_- \cap L)$, where ω is the 2-form (1.4). Hence, the subbundles L_\pm have induced 2-forms ω_\pm^L defined by

$$\begin{aligned} \omega_+^L(F_+l_1, F_+l_2) &= \omega_-^L(F_-l_1, F_-l_2) = \omega(l_1, l_2) = 2g(F_-l_1, F_+l_2) \\ &= -2g(F_+l_1, F_-l_2), \quad l_1, l_2 \in \Gamma L. \end{aligned} \tag{1.10}$$

It is possible to reconstruct L from each of the pairs $(L_{\pm}, \omega_{\pm}^L)$. Namely, with (1.10) one gets

$$\begin{aligned} L &= \{c/F_+(c) \in L_+, g(F_-(c), u) = \frac{1}{2}\omega_+^L(F_+(c), u) \forall u \in L_+\}, \\ L &= \{c/F_-(c) \in L_-, g(F_+(c), v) = -\frac{1}{2}\omega_-^L(F_-(c), v) \forall v \in L_-\}. \end{aligned} \tag{1.11}$$

In particular, if $L_+ = C_+$, L is determined by the 2-form ω_+ on C_+ and it may be called an *almost presymplectic structure*. In this case, the first formula (1.11) shows that L may be identified with the graph of the mapping $(1/2)\sharp_g \circ \flat_{\omega} : C_+ \rightarrow C_-$. If $L_- = C_-$, L is determined by the 2-form ω_- on C_- , it may be called an *almost Poisson structure* and it is the graph of the mapping $-(1/2)\sharp_g \circ \flat_{\omega_-} : C_- \rightarrow C_+$. The condition $L_+ = C_+$ is equivalent with the surjectivity of p_+ , i.e., with $\ker p_+ = C_- \cap L = \{0\}$, and this latter condition also characterizes the almost presymplectic case. Similarly, the almost Poisson case is also characterized by $C_+ \cap L = \{0\}$.

Finally, a *Dirac structure* is an almost Dirac structure which is closed with respect to the bracket $[\cdot, \cdot]_C$. Equivalently, $L \subseteq C$ is a Dirac structure if it is maximal isotropic and $\forall l_a \in \Gamma L$ ($a = 1, 2, 3$) one has

$$g([l_1, l_2]_C, l_3) = 0. \tag{1.12}$$

From the axioms of the Courant algebroids it follows that if L is a Dirac structure then $(L, \rho|_L, [\cdot, \cdot]_C)$ is a Lie algebroid.

In this paper, we will only be interested in the classical Courant case [4]. That is $C = TM \oplus T^*M$ with $\rho(X, \alpha) = X$,

$$g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\beta(X) + \alpha(Y)), \quad F(X, \alpha) = (X, -\alpha), \tag{1.13}$$

therefore,

$$\omega((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) - \beta(X)), \tag{1.14}$$

and with the bracket

$$\begin{aligned} [(X, \alpha), (Y, \beta)] &= ([X, Y], L_X\beta - L_Y\alpha + d(\omega((X, \alpha), (Y, \beta)))) \\ &= ([X, Y], i(X)d\beta - i(Y)d\alpha + \frac{1}{2}d(\beta(X) - \alpha(Y))). \end{aligned} \tag{1.15}$$

In the previous formulas, X, Y are vector fields and α, β are 1-forms on the differentiable manifold M , and the bracket of vector fields is the usual Lie bracket. Notice that $C_+ = TM$, $C_- = T^*M$ and the Courant bracket reduces to zero on T^*M .

Then, a (almost) Dirac structure of $TM \oplus T^*M$ is called a (almost) Dirac structure on the manifold M . By the first formula (1.11), an almost Dirac structure L of M is determined by a generalized distribution $L_+ \subseteq TM$ endowed with a 2-form ω_+^L , namely:

$$L = \{(X, \alpha)/X \in L_+ \ \& \ \alpha|_{L_+} = \flat_{\omega_+^L} X\}. \tag{1.16}$$

By a technical computation, it follows from (1.16) that L is a Dirac structure iff L^+ is integrable and the form ω_+^L is closed on the leaves of L^+ [4]. Accordingly, a Dirac structure on M is equivalent with a generalized foliation with presymplectic leaves where the

presymplectic form depends differentiably of the leaves. If the leaves are symplectic we have a Poisson structure, and if the leaves are the connected components of M we have a presymplectic structure (of a non-constant rank) on M .

Jacobi structures on a manifold M may be seen as a particular case of Dirac structures on $M \times \mathbb{R}$. Namely, a Jacobi structure on M is equivalent with a *Poisson homogeneous structure* on $M \times \mathbb{R}$ (e.g. [7]). We recall that the Poisson structure defined by the bivector field P is homogeneous if there exists a vector field Z such that $P + L_Z P = 0$. This is equivalent with the fact that the Dirac structure $L(P) = \{(\sharp_P \theta, \theta) / \theta \in T^*M\}$ is such that $\forall (X, \theta) \in L(P)$ one has $(X + [Z, X], L_Z \theta) \in L(P)$. The latter property may be attributed to a general Dirac structure L , thus, producing the notion of a general *homogeneous Dirac structure*. A more sophisticated way to see Jacobi structures as Dirac was proposed in [21].

Furthermore, we will be interested in the case where the manifold M is also endowed with a regular foliation \mathcal{F} , and our aim is to extend the notion of \mathcal{F} -coupling from Poisson structures to Dirac structures. Poisson structures coupling with a fibration were studied by Vorobjev [20] then, extended to foliated manifolds in [18]. They proved to be important in the study of the geometry of a Poisson structure in the neighborhood of an embedded symplectic leaf [20]. In [9], Dufour and Wade study a Dirac structure in the neighborhood of a presymplectic leaf and (in our terms) show that the structure is coupling with respect to the fibers of a tubular neighborhood. In the present paper we will define the coupling property of a Dirac structure with respect to an arbitrary foliation and extend Vorobjev’s results. In particular, we will give geometric proofs of some of the results of [9].

Since this paper is a continuation of [18], and in order to avoid repetition, we will use the same notation for everything related with the foliation. In particular, we assume that $\dim M = n$, $\dim \mathcal{F} = p$, $q = n - p$, and we denote by $\Omega^*(M)$, $\mathcal{V}^*(M)$ the spaces of differential forms and multivector fields on M . We will use a normal bundle H , i.e., $TM = H \oplus F$, $F = T\mathcal{F}$, and $T^*M = H^* \oplus F^*$ for the dual bundles $H^* = \text{ann } F$, $F^* = \text{ann } H$ (ann denotes the annihilator space). We will also use the corresponding bigrading of differential forms and multivector fields and the induced decomposition

$$d = d'_{1,0} + d''_{0,1} + \partial_{2,-1} \tag{1.17}$$

of the exterior differential.

2. Coupling Dirac structures

Let (M, \mathcal{F}) be a foliated manifold as described at the end of Section 1. From [18], we recall that a bivector field $P \in \mathcal{V}^2(M)$ is \mathcal{F} -almost coupling via the normal bundle H if $P = P'_{2,0} + P''_{0,2}$, where the indices denote the bidegree, i.e., $P' \in \Gamma \wedge^2 H$, $P'' \in \Gamma \wedge^2 F$. In this case, P satisfies the Poisson condition $[P, P] = 0$ iff the following four conditions hold:

$$\begin{aligned} (L_{\sharp_{P'} \gamma} P')(\alpha, \beta) &= d' \gamma(\sharp_{P'} \alpha, \sharp_{P'} \beta), & (L_{\sharp_{P'} \lambda} P')(\alpha, \beta) &= -\lambda([\sharp_{P'} \alpha, \sharp_{P'} \beta]), \\ (L_{\sharp_{P'} \gamma} P'')(\lambda, \mu) &= 0, & (L_{\sharp_{P''} \nu} P')(\lambda, \mu) &= d'' \nu(\sharp_{P'} \lambda, \sharp_{P''} \mu), \end{aligned} \tag{2.1}$$

where $\alpha, \beta, \gamma \in \Omega^{1,0}(M)$, $\lambda, \mu, \nu \in \Omega^{0,1}(M)$.

Accordingly, the generalization of the almost coupling condition has to ask for a decomposition of the Dirac structure into an F -component and an H -component.

Definition 2.1. Let $L \subseteq TM \oplus T^*M$ be a maximal isotropic subbundle. Denote

$$L_H = L \cap (H \oplus H^*), \quad L_F = L \cap (F \oplus F^*). \tag{2.2}$$

Then, the almost Dirac structure L is \mathcal{F} -almost coupling via H if

$$L = L_H \oplus L_F. \tag{2.3}$$

Therefore, L is almost coupling iff $(Z, \theta) \in L$ is equivalent with $(X, \alpha), (Y, \lambda) \in L$, where $Z = X + Y, \theta = \alpha + \lambda, X \in \Gamma H, Y \in \Gamma F, \alpha \in \Gamma H^*, \lambda \in \Gamma F^*$. Another important observation that follows from (2.3) is that L_H, L_F are maximal isotropic in $H \oplus H^*, F \oplus F^*$, respectively, for the metrics induced by g of (1.13). It follows easily that the bivector field P is \mathcal{F} -almost coupling via H iff the subbundle $L(P)$ satisfies condition (2.3). If $L = L(\tau)$ is an almost presymplectic structure defined by a 2-form τ , almost coupling via H holds iff $\tau = \tau'_{2,0} + \tau''_{0,2}$, where, again, indices denote the bidegree.

In the almost coupling situation, the integrability condition of a maximally isotropic subbundle $L \subseteq TM \oplus T^*M$ extends conditions (2.1).

Proposition 2.1. The \mathcal{F} -almost coupling, almost Dirac structure $L \subseteq TM \oplus T^*M$ is a Dirac structure iff $\forall (X, \alpha) \in \Gamma L_H \forall (Y, \lambda) \in \Gamma L_F$, the following four conditions hold:

$$\begin{aligned} & \sum_{\text{Cycl}(1,2,3)} \{X_1(\alpha_2(X_3)) + \alpha_1([X_2, X_3])\} = 0, \\ & (L_Y \alpha_2)(X_1) + \alpha_1([Y, X_2]) = \lambda([X_1, X_2]), \quad (L_X \lambda_1)(Y_2) + \lambda_2([X, Y_1]) = 0, \\ & ([Y_1, Y_2], i(Y_1) d'' \lambda_2 - i(Y_2) d'' \lambda_1 + d''(\lambda_2(Y_1))) \in L_F. \end{aligned} \tag{2.4}$$

Proof. Since L is isotropic, by (1.9) $\forall (Z_a, \theta_a) \in L (a = 1, 2)$ we have

$$\theta_1(Z_2) + \theta_2(Z_1) = 0. \tag{2.5}$$

Then, since F is involutive and using the decomposition (1.17), the second expression (1.15) of the Courant bracket yields

$$\begin{aligned} [(X_1, \alpha_1), (X_2, \alpha_2)] &= (\text{pr}_H[X_1, X_2], i(X_1) d' \alpha_2 - i(X_2) d' \alpha_1 + d'(\alpha_2(X_1))) \\ &\quad + (\text{pr}_F[X_1, X_2], i(X_1) d'' \alpha_2 - i(X_2) d'' \alpha_1 + d''(\alpha_2(X_1))), \end{aligned} \tag{2.6}$$

$$[(X, \alpha), (Y, \lambda)] = (\text{pr}_H[X, Y], i(X) \partial \lambda - i(Y) d'' \alpha) + (\text{pr}_F[X, Y], i(X) d' \lambda), \tag{2.7}$$

$$\begin{aligned} [(Y_1, \lambda_1), (Y_2, \lambda_2)] &= (0, i(Y_1) d' \lambda_2 - i(Y_2) d' \lambda_1 + d'(\lambda_2(Y_1))) \\ &\quad + ([Y_1, Y_2], i(Y_1) d'' \lambda_2 - i(Y_2) d'' \lambda_1 + d''(\lambda_2(Y_1))), \end{aligned} \tag{2.8}$$

where pr denotes natural projections and the terms are $H \oplus H^*$ and $F \oplus F^*$ components, respectively. In the almost coupling situation, integrability means that these components always belong to L_H, L_F , respectively.

The second term of (2.8) yields the fourth condition (2.4). Since d'' is exterior differentiation along the leaves of \mathcal{F} , this condition is equivalent with the fact that L_F consists of Dirac structures on the leaves of \mathcal{F} ; we will say that L_F is a *leaf-tangent Dirac structure* on (M, \mathcal{F}) .

By maximal isotropy, the $(H \oplus H^*)$ -component of (2.8) belongs to L_H iff $\forall(X, \alpha) \in \Gamma L_H$, the 1-form of the first term of the right-hand side of (2.8) vanishes on X . The result of this evaluation exactly is the third condition (2.4).

The terms of the decompositions (2.7), (2.6) will be treated in a similar way, i.e., using maximal isotropy and evaluations of exterior differentials. The computations show that the condition provided by the $(F \oplus F^*)$ -component of (2.7) is again the third condition (2.4), and the condition provided by the $(H \oplus H^*)$ -component of (2.7) is the second condition (2.4). Then, the condition provided by the $(F \oplus F^*)$ -component of (2.6) is again the second condition (2.4), and the condition provided by the $(H \oplus H^*)$ -component of (2.6) is the first condition (2.4). \square

Remark 2.1. With a few computations, one can see that the Poisson conditions (2.1) for an almost coupling bivector field are exactly the Dirac conditions (2.4) for the subbundle $L(P)$, and in the same order. If only the component L_F is of the almost Poisson type, i.e., the graph of a bivector field $\Pi \in \Gamma \wedge^2 F$, the last formula (2.4) means that Π must be a leaf-tangent Poisson structure of \mathcal{F} [18] and, by putting $Y_a = \sharp_{\Pi} \lambda_a$ ($a = 1, 2$) in the third formula (2.4), the latter becomes

$$(L_X \Pi)(\lambda_1, \lambda_2) = 0. \tag{2.9}$$

Now, on a foliated manifold (M, \mathcal{F}) , a bivector field P is \mathcal{F} -coupling if $\sharp_P(\text{ann } F)$ is a normal bundle of the foliation \mathcal{F} . In order to extend this notion, with any maximal isotropic subbundle $L \subseteq TM \oplus T^*M$, we associate the possibly non-differentiable, generalized distribution of M defined by

$$H_x(L, \mathcal{F}) = \{Z \in T_x M / \exists \alpha \in \text{ann } F_x \ \& \ (Z, \alpha) \in L\} \quad (x \in M). \tag{2.10}$$

Then, state the following definition.

Definition 2.2. The almost Dirac structure L is \mathcal{F} -coupling if the distribution $H = H(L, \mathcal{F})$ is normal to the foliation \mathcal{F} at each point $x \in M$.

Examples of coupling (integrable) Dirac structures will be shown in the next section. In particular, we will see that, for any Dirac structure L and any embedded presymplectic leaf S of L , there exists a tubular neighborhood U of S such that L is coupling with the tubular fibers on U .

Proposition 2.2. *If the subbundle L is \mathcal{F} -coupling $\forall x \in M$, L_x is \mathcal{F} -almost coupling at x via $H_x = H_x(L, \mathcal{F})$. Furthermore, the $(H \oplus H^*)$ -component of L_x is the graph of a mapping*

$b_{\sigma_x} : H_x \rightarrow H_x^*$ defined by some $\sigma_x \in \wedge^2(\text{ann } F_x)$, and the $(F \oplus F^*)$ -component of L_x is the graph of a mapping $\sharp_{\Pi_x} : F_x^* \rightarrow F_x$ defined by some $\Pi_x \in \wedge^2 F_x$. Moreover, $H = H(L, \mathcal{F})$ is a differentiable, normal bundle of \mathcal{F} such that L is \mathcal{F} -almost coupling via H , and the global cross-sections $\sigma \in \Gamma \wedge^2(\text{ann } F)$, $\Pi \in \Gamma \wedge^2 F$ are differentiable.

Proof. The following considerations are at a fixed point $x \in M$, which we do not include in the notation. With $H = H(L, \mathcal{F})$ as the normal space of \mathcal{F} at x , take $(Z, \theta) \in L$ and decompose $Z = X + Y$, $X \in H$, $Y \in F$. By the definition of H , $\exists \alpha \in H^*$ such that $(X, \alpha) \in L$, and we get a decomposition

$$(Z, \lambda) = (X, \alpha) + (Y, \theta - \alpha), \tag{2.11}$$

where the terms belong to L . Then, $\forall X' \in H$ with a corresponding covector α' such that $(X', \alpha') \in L$, (2.5) implies

$$(\theta - \alpha)(X') = -\alpha'(Y) = 0.$$

Hence $\theta - \alpha \in F^*$ and (2.11) implies the almost coupling property (2.3) at x .

Furthermore, if $(X, \alpha), (X, \alpha') \in L$, where $\alpha, \alpha' \in H^*$, we get $(0, \alpha' - \alpha) \in L$ and the isotropy of L together with the coupling hypothesis imply $\alpha' = \alpha$. Therefore, $\forall X \in H$, the covector $\alpha \in \text{ann } F$ such that $(X, \alpha) \in L$ is unique and L_H is the graph of a morphism b_σ , $\sigma \in \wedge^2 H^*$. Notice also that the uniqueness of α is equivalent with

$$L \cap \text{ann } F = \{0\}. \tag{2.12}$$

On the other hand, the definition of H implies $L \cap F \subseteq H$, therefore, in the coupling case, $L \cap F = \{0\}$ and (see Section 1) L_F must be of the almost Poisson type, whence the existence of Π .

Finally, we will prove the differentiability of the distribution $H(L, \mathcal{F})$. For this purpose, let us consider the subspaces

$$\tilde{H}(L, \mathcal{F}) = \{(Z, \alpha) \in L / \alpha \in \text{ann } F\} = L \cap [TM \oplus (\text{ann } F)] \tag{2.13}$$

at each point of M . Then, $\ker p_+|_{\tilde{H}(L, \mathcal{F})} = L \cap \text{ann } F$ and

$$H(L, \mathcal{F}) = p_+(\tilde{H}(L, \mathcal{F})) \approx \tilde{H}(L, \mathcal{F}) / (L \cap \text{ann } F). \tag{2.14}$$

In the coupling case, because of (2.12), $p_+|_{\tilde{H}(L, \mathcal{F})}$ is an isomorphism and we are done if we prove the differentiability of $\tilde{H}(L, \mathcal{F})$. In a neighborhood of a point x , let $\{\alpha^a\}$, $\{\lambda^u\}$ and $\{(Z_i, \theta_i = \theta_{ia}\alpha^a + \theta_{iu}\lambda^u)\}$ be differentiable, local bases of H^* , F^* and L , respectively. Then,

$$\tilde{H}(L, \mathcal{F}) = \{(\xi^i Z_i, \xi^i \theta_i) / \theta_{iu} \xi^i = 0\}$$

(here and in the whole paper we use the Einstein summation convention), and we see that $\tilde{H}(L, \mathcal{F})$ is locally generated by fundamental solutions of a linear, homogeneous system of equations with differentiable coefficients. But, if the rank of the latter is constant (and under the coupling hypothesis the rank is $n - q$), differentiable, fundamental solutions exist. Of

course, the differentiability of H also implies the differentiability of the 2-form σ and of the bivector field Π . \square

Hereafter, in the coupling situation we will use only $H(L, \mathcal{F})$ as the normal bundle and shortly denote it by H . The coupling situation is interesting precisely because it provides a canonical normal bundle of \mathcal{F} .

Proposition 2.3. *An \mathcal{F} -coupling, almost Dirac structure $L \subseteq TM \oplus T^*M$ is equivalent with a triple (H, σ, Π) , where H is a normal bundle of the foliation \mathcal{F} , $\sigma \in \Gamma \wedge^2(\text{ann } F)$ and $\Pi \in \Gamma \wedge^2 F$.*

Proof. We have already derived the triple from the subbundle L . Such a triple is called a system of *geometric data* [20,9]. Conversely, if the geometric data are given, we reconstruct $L = L_H \oplus L_F$ by defining L_H as the graph of b_σ and L_F as the graph of $\sharp\Pi$. In other words, we have

$$L = \{(X, b_\sigma X) + (\sharp\Pi\lambda, \lambda) / X \in H, \lambda \in F^*\}. \quad \square \tag{2.15}$$

Corollary 2.1. *On (M, \mathcal{F}) , the almost Dirac structure L is \mathcal{F} -coupling iff*

$$L \cap (F \oplus \text{ann } F) = \{0\}. \tag{2.16}$$

Proof. Condition (2.16) is an immediate consequence of (2.15). Conversely, (2.16) implies $H \cap F = \{0\}$ and $H \approx \tilde{H}$. On the other hand, by looking at dimensions, (2.16) also implies

$$L \oplus (F \oplus \text{ann } F) = TM \oplus T^*M, \tag{2.17}$$

therefore,

$$L + (TM \oplus \text{ann } F) = TM \oplus T^*M,$$

and from (2.13) we get $\dim H = \dim \tilde{H} = q$. \square

Remark 2.2. In the integrable case, formula (2.17) shows a new structure of para-Hermitian Courant algebroid on $TM \oplus T^*M$, the double of a Lie bialgebroid $(L, F \oplus \text{ann } F)$.

Remark 2.3. An almost Poisson structure $L(P)$ defined by the bivector field P is coupling iff there exists a normal bundle H of the foliation \mathcal{F} that yields $P = P'_{2,0} + P''_{0,2}$, where P' is non-degenerate [20]. In the case of an almost presymplectic structure $L(\tau)$ defined by a 2-form τ , $H(L(\tau), \mathcal{F})$ is the τ -orthogonal distribution of F and the coupling condition holds iff the former is a complementary distribution of the latter. Equivalently, $L(\tau)$ is \mathcal{F} -coupling iff there exists a normal bundle H that yields a decomposition

$$\tau = \tau'_{2,0} + \tau''_{0,2}, \tag{2.18}$$

where τ'' is non-degenerate.

Remark 2.4. One can also define the notion of an \mathcal{F} -coupling Dirac structure in a dual way. Namely, for any almost Dirac structure $L \subseteq TM \oplus T^*M$ of the foliated manifold (M, \mathcal{F}) ,

we may define the *generalized codistribution* (a field of subspaces of the fibers of T^*M with a varying dimension)

$$K_x^* = K_x^*(L, \mathcal{F}) = \{\theta \in T_x^*M / \exists Y \in F_x \ \& \ (Y, \theta) \in L_x\} \quad (x \in M). \tag{2.19}$$

K^* may not be differentiable, i.e., it may not have local generators defined by differentiable 1-forms. Then, it follows that L is \mathcal{F} -coupling iff

$$T^*M = (\text{ann } F) \oplus K^*. \tag{2.20}$$

Indeed, by dualizing the proof of [Proposition 2.2](#), we see that condition (2.20) also obliges L to be of the form (2.15). Notice also that, in the coupling case, the decomposition (2.20) is the dual of $TM = H \oplus F$ for H given by (2.10).

The integrability conditions of a coupling Dirac structure may also be expressed by means of the associated geometric data like in the Poisson case [20,18].

Proposition 2.4. *An \mathcal{F} -coupling almost Dirac structure $L \subseteq TM \oplus T^*M$ of a foliated manifold (M, \mathcal{F}) is a Dirac structure iff its associated geometric data (H, σ, Π) satisfy the following conditions:*

- (i) Π is a leaf-tangent Poisson structure on (M, \mathcal{F}) , i.e., its restriction to each leaf is a Poisson structure of the leaf;
- (ii) $d'\sigma = 0$, equivalently, $d\sigma(X_1, X_2, X_3) = 0, \forall X_1, X_2, X_3 \in \Gamma H$;
- (iii) for any projectable (to the space of leaves of \mathcal{F}) vector fields $X_1, X_2 \in \Gamma_{\text{pr}} H$ (pr denotes projectability) one has

$$\text{pr}_F[X_1, X_2] = \sharp_{\Pi}(d''(\sigma(X_1, X_2)));$$

- (iv) for any projectable vector field $X \in \Gamma_{\text{pr}} H$ one has $L_X \Pi = 0$.

Proof. Condition (i) is the equivalent of the fourth formula (2.4) if $L_F = L(\Pi)$ is the graph of Π . In the coupling case, if we put $\alpha_a = \flat_{\sigma} X_a$ ($a = 1, 2, 3$) in the first formula (2.4), we get condition (ii). The similar replacement of the forms α in the second formula (2.4) puts the latter into the form

$$(L_Y \sigma)(X_1, X_2) = -\lambda([X_1, X_2]), \quad \forall Y = \sharp_{\Pi} \lambda, \forall \lambda \in F^*, \forall X_1, X_2 \in \Gamma H. \tag{2.21}$$

Since this condition is invariant by multiplication of the arguments X by any $f \in C^{\infty}(M)$, it suffices to ask (2.21) for projectable arguments. But, $X \in \Gamma_{\text{pr}} H$ iff $[Y, X] \in \Gamma F, \forall Y \in \Gamma F$ [15], and we see that (2.21) is equivalent with

$$(\sharp_{\Pi} \lambda)(\sigma(X_1, X_2)) = -\lambda([X_1, X_2]),$$

which exactly is condition (iii) of the proposition. Similarly, it suffices to use a projectable argument X in the third formula (2.4). Then, the third formula (2.4) becomes $([X, \sharp_{\Pi} \lambda], L_X \lambda) \in \Gamma L(\Pi)$, which is equivalent with condition (iv). \square

Remark 2.5. Conditions (i)–(iv) of [Proposition 2.4](#) are the same as Vorobjev’s conditions [20,18] of the Poisson case, except for the fact that the 2-form σ may degenerate. In [9]

these conditions were included by definition. In the presymplectic case, where the structure is defined by the closed 2-form τ of (2.18) with a non-degenerate component τ'' , condition (i) means that τ'' defines symplectic structures on the leaves of \mathcal{F} , and conditions (ii)–(iv) become

$$d'\tau' = 0, \quad d''(\tau'(X_1, X_2)) = \flat_{\tau''}(\text{pr}_F[X_1, X_2]), \quad L_X\tau'' = 0 \tag{2.22}$$

$\forall X, X_1, X_2 \in \Gamma_{\text{pr}} H$. It is easy to see that these conditions are equivalent with

$$d''\tau'' = 0, \quad d'\tau' = 0, \quad d''\tau' + \partial\tau'' = 0, \quad d'\tau'' = 0, \tag{2.23}$$

which are the homogeneous components of $d\tau = 0$.

3. Dirac structures near a presymplectic leaf

We continue to use the notation of the previous sections. A presymplectic leaf of a Dirac structure L of a differentiable manifold M is an integral submanifold of the distribution L_+ defined in Section 1. In [9], it was proven that, in a tubular neighborhood of an embedded presymplectic leaf, any Dirac structure L is coupling with respect to the fibers of the tubular structure. This result, which extends the similar one in Poisson geometry [20], describes the geometry of a Dirac structure near an embedded presymplectic leaf. Below, we give an invariant proof of this result.

Proposition 3.1. *Let L be a Dirac structure on the foliated manifold (M, \mathcal{F}) . Assume that L has a presymplectic leaf S such that $T_S M = TS \oplus F|_S$. Then, there exists an open neighborhood U of S in M such that $L|_U$ is coupling with respect to $\mathcal{F} \cap U$.*

Proof. We will refer to bidegrees defined by the decomposition $T_S M = TS \oplus F|_S$, where we know that $TS = L_+|_S$. Then, for all $X \in TS$, there exists a covector $\alpha \in T^*S$ of bidegree $(1, 0)$ equal to $\flat_{\omega_+^L} X$ on TS , and (1.16) shows that $(X, \alpha) \in L$. The conclusion is that $L|_S$ is the presymplectic structure of S and $H(L, \mathcal{F})|_S = TS$, therefore, L is \mathcal{F} -coupling along S . By Corollary 2.1, this is equivalent with

$$[L + (F \oplus \text{ann } F)]_S = T_S M \oplus T_S^* M, \tag{3.1}$$

and, since its left hand side is differentiable, (3.1) also holds on an open neighborhood U of S . \square

Corollary 3.1 (Dufour–Wade [9]). *Assume that the Dirac structure L of a manifold M has an embedded leaf S . Then, on a sufficiently small tubular neighborhood U of S , with the foliation \mathcal{F} defined by the fibers of the tubular structure, L may be put under the form (2.15), where $(H, \sigma \in \Gamma \wedge^2(\text{ann } F), \Pi \in \Gamma \wedge^2 F)$ is a triple of geometric data that satisfies the integrability conditions (i) – (iv) of Proposition 2.4, and Π is a Poisson structure which vanishes on S .*

Remark 3.1 (Dufour–Wade [9]). The previous corollary implies the fact that all the presymplectic leaves of a Dirac structure have the same parity. Indeed, with the notation of the

corollary, at a point near S , L_+ is a direct sum of a subspace of dimension $\dim S$ and a subspace tangent to a symplectic leaf of Π , which has an even dimension.

Following [9], we say that Π is the *transversal Poisson structure* of the leaf S . The following proposition shows that the transversal structure is essentially unique.

Proposition 3.2 (Dufour–Wade [9]). *The transverse Poisson structure of an embedded presymplectic leaf of a Dirac structure is unique up to Poisson equivalence.*

Proof. We get two transversal structures Π_1, Π_2 if we use two tubular neighborhoods U_1, U_2 . The isotopy of the latter [10] yields a leaf-preserving diffeomorphism $\Phi : U_1 \rightarrow U_2$, which may be seen as the composition of maps in the flows of projectable vector fields X on U_1 . But, (2.9) implies $L_X \Pi = 0$ for all the projectable vector fields. Hence, by the integrability condition (iv) of Proposition 2.4, $\Phi_* \Pi_1 = \Pi_2$ (after shrinking the tubular neighborhoods if necessary). \square

In what follows we consider the Vorobjev–Poisson structure [20,18] in the case of an embedded, presymplectic leaf S of a Dirac structure L .

From the general result on Lie algebroids given by Theorem 2.1 of [11], it follows that the vector bundle $L|_S$ has a well defined induced structure of a transitive Lie algebroid over S with the anchor and bracket defined by

$$\begin{aligned} \rho(X, \theta) &= X, \\ [(X_1, \theta_1), (X_2, \theta_2)]_S &= ([\tilde{X}_1, \tilde{X}_2], L_{\tilde{X}_1} \tilde{\theta}_2 - L_{\tilde{X}_2} \tilde{\theta}_1 - d(\tilde{\theta}_2(\tilde{X}_1)))|_S, \end{aligned} \tag{3.2}$$

where all the pairs (X, θ) belong to $L|_S$ and $(\tilde{X}, \tilde{\theta})$ are arbitrary extensions of (X, θ) from S to M . The existence of such extensions follows from the fact that S is an embedded submanifold, and the independence of the bracket (3.2) of the choice of the extensions follows easily if the extensions are expressed via a local basis of L and the axioms of a Lie algebroid are used [11].

Accordingly, $\mathbb{G} = \ker \rho$ is a bundle of Lie algebras such that

$$0 \rightarrow \mathbb{G} \xrightarrow{\subseteq} L|_S \xrightarrow{\rho} TS \rightarrow 0 \tag{3.3}$$

is an exact sequence with projections p onto S , while S is endowed with the presymplectic (closed) 2-form $\varpi = \omega_+^L$.

As in [20,18], each splitting $\gamma : TS \rightarrow L|_S$ produces geometric data $(\mathcal{H}, \sigma, \mathbb{L})$ on the total space of the dual bundle \mathbb{G}^* with the foliation by fibers \mathcal{V} . Namely:

- (i) \mathcal{H} is the horizontal bundle of the dual of the connection defined on \mathbb{G} by the formula

$$\nabla_X \eta = [\gamma(X), \eta]_S, \quad X \in \Gamma TS, \quad \eta \in \Gamma \mathbb{G}, \tag{3.4}$$

- (ii) σ is the 2-form evaluated, at $z \in \mathbb{G}^*$, on horizontal lifts $\mathcal{X}_1, \mathcal{X}_2$ of $X_1, X_2 \in \Gamma TS$, by

$$\sigma_z(\mathcal{X}_1, \mathcal{X}_2) = \varpi_{p(z)}(X_1, X_2) - z([\gamma(X_1), \gamma(X_2)] - \gamma[X_1, X_2]), \tag{3.5}$$

- (iii) \mathbb{L} is the family of Lie–Poisson structures of the fibers of \mathbb{G}^* .

The only difference between this situation and that of Vorobjev [20] is that ϖ may degenerate. But, the arguments and computations of Vorobjev’s case, as described in [18] are still valid, and they show that the triple $(\mathcal{H}, \sigma, \mathbb{L})$ satisfies the integrability conditions i)-iv) of Proposition 2.4. Therefore, there exists a corresponding coupling Dirac structure $\mathcal{L}(S, \gamma)$ on the manifold \mathbb{G}^* , and we call it an *associated Dirac structure of L along S* .

Let νS be a normal bundle of the leaf S ($T_S M = \nu S \oplus TS$). Then, the reconstruction formula (1.16) shows that

$$L|_S = \{(X, \flat_{\varpi} X + \lambda) / X \in TS, \lambda \in \nu^* S\}, \tag{3.6}$$

where $\flat_{\varpi} X$ is the 1-form defined by

$$(\flat_{\varpi} X)(Z) = \begin{cases} \varpi(X, Z), & \text{if } Z \in TS, \\ 0, & \text{if } Z \in \nu S, \end{cases} \tag{3.7}$$

and we have a splitting γ given by

$$\gamma(X) = (X, \flat_{\varpi} X). \tag{3.8}$$

On the other hand, (3.6) shows that we may identify the bundle \mathbb{G}^* with νS . Namely, we have $\mathbb{G} = \nu^* S = \text{ann } TS$, and $\Theta \in \text{Hom}(\text{ann } TS, \mathbb{R})$ identifies with $Y \in \nu S$ defined by $\lambda(Y) = \Theta(\lambda) \forall \lambda \in \text{ann } TS$. We will say that the Dirac structure associated with L by the splitting (3.8) is the *associated, normal Dirac structure $\mathcal{L}(S, \nu S)$* .

We want to find a convenient local coordinate expression of $\mathcal{L}(S, \nu S)$. For this purpose, around the points of S and after the choice of the normal bundle νS , we take local coordinates (x^u, y^a) , where a (and similar indices b, c) takes the values $1, \dots, \text{codim } S$ and u (and similar indices v, w) takes the values $1, \dots, \text{dim } S$, such that S is locally defined by the equations $y^a = 0$ and

$$\begin{aligned} TS &= \text{span} \left\{ \frac{\partial}{\partial x^u} \Big|_{y^b=0} \right\}, & \nu S &= \text{span} \left\{ \frac{\partial}{\partial y^a} \Big|_{y^b=0} \right\}, & T^* S &= \text{span}\{dx^u \Big|_{y^b=0}\}, \\ \nu^* S &= \text{span}\{dy^a \Big|_{y^b=0}\}. \end{aligned} \tag{3.9}$$

Then, Theorem 3.2 of [9] tells us that the Dirac structure L has local bases of the form

$$\begin{aligned} \mathcal{H}_u &= \left(\frac{\partial}{\partial x^u} + A_u^b(x, y) \frac{\partial}{\partial y^b}, \alpha_{uv}(x, y) dx^v \right), \\ \mathcal{Y}^a &= \left(B^{ab}(x, y) \frac{\partial}{\partial y^b}, dy^a - A_v^a(x, y) dx^v \right), \end{aligned} \tag{3.10}$$

where $\alpha_{uv}(x, 0)$ are the components of the 2-form ϖ and

$$A_u^b(x, 0) = 0, \quad B^{ab}(x, 0) = 0. \tag{3.11}$$

Remark 3.2. As a matter of fact the proof given in [9] holds to show that, for any maximal isotropic subspace $L \subseteq W \oplus W^*$ where W is an arbitrary vector space, there are bases of the form

$$(l_u + A_u^b f_b, \alpha_{uv} \lambda^v), \quad (B^{ab} f_b, \varphi^a - A_v^a \lambda^v), \tag{3.12}$$

where (l_u) is a basis of L_+ , (f_a) is a basis of an arbitrary complement of L_+ , (λ^u) is the dual basis of (l_u) and (φ^a) is the dual basis of (f_a) .

It follows that the local basis of the Lie algebra bundle $\mathbb{G} \approx \nu^*S$ given in (3.9) may be seen as $(\mathcal{V}^a|_{y^b=0})$ and the local basis of $\gamma(TS)$ is $(\mathcal{H}_u|_{y^b=0})$. Using (3.11), which implies the vanishing of the derivatives of the same functions with respect to x^u on S , and the closedness of the 2-form ϖ , we get for the brackets of the elements of these bases the expressions

$$\begin{aligned}
 [\mathcal{V}^a, \mathcal{V}^b]_S &= \left(\frac{\partial B^{ab}}{\partial y^c} \mathcal{V}^c \right)_{y^b=0}, & [\mathcal{H}_u, \mathcal{V}^a]_S &= \left(\frac{\partial A_u^a}{\partial y^c} \mathcal{V}^c \right)_{y^b=0}, \\
 [\mathcal{H}_u, \mathcal{H}_v]_S &= \left(\frac{\partial \alpha_{uv}}{\partial y^c} \mathcal{V}^c \right)_{y^b=0}.
 \end{aligned}
 \tag{3.13}$$

Accordingly, as in [18], we get the following expressions of the geometric data that define the Dirac structure $\mathcal{L}(S, \nu S)$

$$\mathcal{H} = \text{span} \left\{ \mathcal{X}_u = \frac{\partial}{\partial x^u} + \frac{\partial A_u^a}{\partial y^c} \eta^c \frac{\partial}{\partial \eta^a} \right\}_{y^b=0}, \tag{3.14}$$

$$\sigma(\mathcal{X}_u, \mathcal{X}_v) = \alpha_{uv}(x, 0) - \left(\frac{\partial \alpha_{uv}}{\partial y^a} \right)_{y^b=0} \eta^a, \tag{3.15}$$

$$\mathbb{L}^{ab} = \left(\frac{\partial B^{ab}}{\partial y^c} \right)_{y^b=0} \eta^c, \tag{3.16}$$

where (η^a) are fiber coordinates on νS .

If we use the previous formulas and (2.15) we get a canonical, local basis of type (3.10) for $\mathcal{L}(S, \nu S)$ namely,

$$\begin{aligned}
 & \left(\left[\frac{\partial}{\partial x^u} + \frac{\partial A_u^a}{\partial y^c} \eta^c \frac{\partial}{\partial \eta^a} \right]_{y^b=0}, \left[\left(\alpha_{uv}(x, 0) - \left(\frac{\partial \alpha_{uv}}{\partial y^a} \right) \eta^a \right) dx^v \right]_{y^b=0} \right), \\
 & \left(\left[\frac{\partial B^{ab}}{\partial y^c} \eta^c \frac{\partial}{\partial \eta^b} \right]_{y^b=0}, \left[d\eta^a - \frac{\partial A_u^a}{\partial y^c} \eta^c dx^u \right]_{y^b=0} \right).
 \end{aligned}
 \tag{3.17}$$

From (3.17), we see that S , seen as the zero section of νS endowed with the 2-form ϖ , is a presymplectic leaf of $\mathcal{L}(S, \nu S)$. Moreover, we see that the structure $\mathcal{L}(S, \nu S)$ along S is a linear approximation of the Dirac structure L along S .

For a Poisson structure P , Vorobjev proved that the Poisson structures $\mathcal{L}(S, \nu S)$ defined by different normal bundles νS are equivalent in neighborhoods of S [20,18]. We will see below why his proof does not apply in the general Dirac case, and indicate a particular case where it works.

The choice of the normal bundle νS is equivalent with the definition of a projection epimorphism $\Pi : T_S M \rightarrow TS$ ($\Pi^2 = \Pi$) namely, $\nu S = \ker \Pi$. A second normal bundle $\nu' S$ corresponds to a second epimorphism $\Pi' : T_S M \rightarrow TS$, and $\Pi_t = (1 - t)\Pi + t\Pi'$,

$t \in \mathbb{R}$, defines a homotopy between the normal bundles $\nu S(t = 0)$ and $\nu' S(t = 1)$ by a family of normal bundles $\nu_t S$.

Furthermore, there exists a bundle isomorphism $\Phi_1 : \nu S \rightarrow \nu' S$ defined by

$$\Phi_1(Y) = Y - \Pi'(Y), \quad Y \in \nu S.$$

Similarly, we have isomorphisms

$$\Phi_t = t\Phi_1 - (1 - t)\text{Id} : \nu S \rightarrow \nu_t S.$$

On each $\nu_t S$ we have a Dirac structure $\mathcal{L}_t(S, \nu_t S)$, and we may pullback all these structures to νS by Φ_t^{-1} .

On \mathbb{G}^* seen as the fixed normal bundle νS , this exactly provides the homotopy considered by Vorobjev [20,18] between the Dirac structures $\mathcal{L}(S, \gamma), (\Phi_1^{-1})_*(\mathcal{L}(S, \gamma'))$, where γ, γ' are the splittings of the exact sequence (3.3) associated with $\nu S, \nu' S$ by (3.8). Indeed, formula (3.8) implies that the \mathbb{G} -valued difference form $\phi = \gamma' - \gamma$ is given by

$$\phi(X) = (i(X)\varpi) \circ \Pi' \in \text{ann } TS \quad (X \in TS), \tag{3.18}$$

and, if we write the same form for Π_t instead of Π' we get the form $t\phi$.

In the Poisson case, Vorobjev’s proof is based on the 1-form

$$\psi_Y(\mathcal{X}) = \langle \phi(X), Y \rangle = \varpi(X, \Pi'Y), \quad Y \in \nu S = \mathbb{G}^*, \quad X \in TS, \tag{3.19}$$

defined on the total space of the bundle νS , where \mathcal{X} is the horizontal lift of X by the connection (3.4). The horizontal, time dependent, tangent vector field Ξ_t of νS that satisfies the condition $\flat_{\sigma_t} \Xi_t = -\psi$ has a flow which, at time 1, yields the required equivalence [20,18].

In the general Dirac case, the form ψ exists but, the vector field Ξ_t may not exist since the form σ_t is no more non-degenerate.

Let us refer to the particular case of a Dirac structure L such that the field of planes $K = L \cap TM$ has a constant dimension k . Since L is closed by the Courant bracket, K is involutive, therefore, tangent to a foliation \mathcal{K} , the leaves of which are submanifolds of the presymplectic leaves of L . A Dirac structure with the previous property will be called *locally reducible*, because if the stronger property that the leaves of \mathcal{K} are the fibers of a fibration holds the Dirac structure is *reducible* [13]. For a locally reducible Dirac structure the local coordinates (x^u) used to get the canonical bases of L given by (3.10) may be taken under the form (x^e, z^s) ($s = 1, \dots, k, e = 1, \dots, \dim S - k$), where $x^e = \text{const}$. define the leaves of \mathcal{K} and (z^s) are coordinates along these leaves.

Proposition 3.3. *Let L be a locally reducible Dirac structure, $x \in M$ and S the presymplectic leaf through x . Then, there exists a neighborhood of x where L has local bases of the form*

$$\begin{aligned} \mathcal{Z}_s &= \left(\frac{\partial}{\partial z^s}, 0 \right), & \mathcal{H}_e &= \left(\frac{\partial}{\partial x^e} + A_e^b(x, y) \frac{\partial}{\partial y^b}, \alpha_{ef}(x, y) dx^f \right), \\ \mathcal{V}^a &= \left(B^{ab}(x, y) \frac{\partial}{\partial y^b}, dy^a - A_f^a(x, y) dx^f \right), \end{aligned} \tag{3.20}$$

the coefficients A, B vanish at $y^a = 0$ and $\alpha_{ef}(x, 0)$ are the local components of the presymplectic 2-form ϖ of S .

Proof. By the definition of K and of the local coordinates that we use, $\mathcal{Z}_s \in L$ and must be g -orthogonal to the vectors of the basis (3.10). This happens iff (3.10) are of the form (3.20), where, a priori, the remaining coefficients may depend on all the coordinates (x, y, z) . But, since L is closed by Courant brackets, $[\mathcal{Z}_s, \mathcal{H}_e], [\mathcal{Z}_s, \mathcal{V}^i]$ must be g -orthogonal to $\mathcal{Z}_s, \mathcal{H}_e$ and this implies

$$\frac{\partial \alpha_{ef}}{\partial z^s} = 0, \quad \frac{\partial A_e^b}{\partial z^s} = 0, \quad \frac{\partial B^{ab}}{\partial z^s} = 0. \quad \square$$

Now, it is easy to prove the following proposition.

Proposition 3.4. *Let L be a reducible Dirac structure and S an embedded presymplectic leaf of L . Then, the associated, normal Dirac structures defined by any two normal bundles of S are equivalent.*

Proof. If $\psi : M \rightarrow M/K$ is the reducibility fibration, the vectors $\mathcal{H}_e, \mathcal{V}^i$ of the canonical bases (3.20) are ψ -projectable and their projections define a Poisson structure Λ on M/K for which S/K is an embedded symplectic leaf. (This is a well known result [4,13]. The structure Λ is Poisson because, on $S, K = \ker \varpi$, hence, the matrix (α_{ef}) of (3.20) has a maximal rank.) Moreover, the projections of the vectors \mathcal{V}^i of (3.20) yield a normal bundle of $S/K \subseteq M/K$ and, because the computation of the Courant brackets of $\mathcal{H}_e, \mathcal{V}^i$ on M and on M/K is the same, the associated, normal Dirac structure of L along S and that of the projected Poisson structure Λ of M/K along S/K correspond each to the other by ψ . If we act like that for two normal bundles of S , we get two associated, normal Poisson structures of Λ , which are Poisson equivalent by Vorobjev’s theorem [20,18]. This Poisson equivalence lifts to an equivalence of the associated, normal Dirac structures of L . (The triples that define the associated structures on M and on M/K have the same local coordinate expressions with respect to the bases (3.20) and their projections.) \square

4. Submanifolds of Dirac manifolds

In this section we will show that the almost-coupling and coupling conditions are also significant for single submanifolds N^p of a Dirac manifold (M^n, L) (a differentiable manifold M with a fixed Dirac structure L), as opposed to a whole foliation \mathcal{F} . For simplicity, we will assume that the submanifold is embedded. Where possible, we will continue to use the notation of the previous sections.

We begin by recalling that Dirac structures may be both pulled back and pushed forward pointwisely (e.g., see [4,2]). If $\phi : N \rightarrow M$ is a differentiable mapping between arbitrary manifolds and $L(M)$ is a Dirac structure on M then $\forall x \text{ inf } N$,

$$(\phi^*(L(M)))_x = \{(Z, \phi^* \alpha) / Z \in T_x M, \alpha \in T_{\phi(x)}^* M, (\phi_{*,x} Z, \alpha) \in L(M)_{\phi(x)}\} \quad (4.1)$$

is a maximal isotropic subspace of $T_x N \oplus T_x^* N$

On the other hand, if we have a Dirac structure $L(N)$ on N and $x \in N$, we have the maximal isotropic subspace

$$\begin{aligned}
 (\phi_*(L(N)))_{\phi(x)} &= \{(\phi_{*,x}Z, \alpha)/\alpha \in T_{\phi(x)}^*M, (Z, \phi_x^*\alpha) \in L(N)_x\} \\
 &\subseteq T_{\phi(x)}M \oplus T_{\phi(x)}^*M.
 \end{aligned}
 \tag{4.2}$$

Generally, these pointwise operations do not yield differentiable subbundles. If differentiable Dirac structures $L(N), L(M)$ are related by (4.1), ϕ is called a *backward Dirac map*, and if the relation is (4.2) ϕ is a *forward Dirac map* [2]. If ϕ is the embedding $\iota : N^p \hookrightarrow M^n$ of a submanifold and if $L(N) = \iota^*(L(M))$ is differentiable, $L(N)$ must be integrable, and it defines a Dirac structure on N [4]. Indeed, $L(N)$ is equivalent in the sense of (1.11) with the field of planes

$$\begin{aligned}
 L(N)_{+,x} &= \{Z \in T_xN/\exists\alpha \in T_x^*M(Z, \alpha) \in L(M)_x\} = L(M)_{+,x} \cap T_xN \\
 &= T_x(S(L(M))) \cap T_xN
 \end{aligned}
 \tag{4.3}$$

(S denotes presymplectic leaves), endowed with the 2-form $\omega_+^{L(N)}$ induced by $\omega_+^{L(M)}$. Obviously, if (4.3) is a differentiable distribution, it is integrable and $\omega_+^{L(N)}$ is closed. In what follows, if the induced Dirac structure $L(N) = \iota^*(L)$ is differentiable, we will call N a *proper submanifold* of (M, L) .

Along the submanifold N of (M, L) , we have the field of planes

$$\begin{aligned}
 K(N) &= \ker \flat_{\omega_{S(N)}} = L(N) \cap TN = \{(Z, 0)/Z \in TN \ \& \ \exists\alpha \in \text{ann } TN, (Z, \alpha) \in L\} \\
 &= \text{pr}_{TN}(L \cap (TN \oplus \text{ann } TN)),
 \end{aligned}
 \tag{4.4}$$

the kernel of the induced structure $L(N)$. If L is defined by a Poisson bivector field $P \in \mathcal{V}^2(M)$, a proper submanifold with kernel zero has an induced structure $L(N)$ provided by a Poisson bivector field $\Pi \in \mathcal{V}^2(N)$. Such submanifolds were studied in [5] under the name of *Poisson–Dirac submanifolds*.

On the other hand, if $\iota : N \hookrightarrow M$ is a submanifold of (M, L) and νN is a normal bundle, we may use the push forward construction (4.2) along N and get a maximally isotropic subbundle $\text{pr}_N L \subseteq TN \oplus T^*N$ given by

$$\text{pr}_N L = \{(\text{pr}_{TN} Z, \text{pr}_{T^*N} \alpha)/(Z, \alpha) \in L|_N\},
 \tag{4.5}$$

where the involved projections are those of the decomposition $T_N M = \nu N \oplus TN$. Obviously, this subbundle is differentiable. A pair $(N, \nu N)$ is called a *normalized submanifold* [17], and, along it, one has *adapted local coordinates* of M like those that appear in (3.9). In the present case, if $\text{pr}_N L$ of (4.5) is also integrable, it yields a Dirac structure and we will say that $(N, \nu N, \text{pr}_N L)$ is a *submersed submanifold* of (M, L) .

Now, like in Definition 2.1, we define

Definition 4.1. The pair $(N, \nu N)$ is a *properly normalized submanifold* of the Dirac manifold (M, L) if

$$L|_N = L_{\nu N} \oplus L_{TN},
 \tag{4.6}$$

where

$$L_{\nu N} = L \cap (\nu N \oplus \nu^* N), \quad L_{TN} = L \cap (TN \oplus T^* N). \tag{4.7}$$

Proposition 4.1. *A properly normalized submanifold $(N, \nu N)$ of a Dirac manifold (M, L) is simultaneously proper and submersed and has the differentiable, Dirac structure*

$$L(N) = \iota^*(L) = \text{pr}_N L, \tag{4.8}$$

where the projection is defined by the decomposition $T_N M = \nu N \oplus TN$.

Proof. It is easy to understand that condition (4.6) is equivalent with

$$(Z, \theta) \in L|_N \Rightarrow (\text{pr}_{TN} Z, \text{pr}_{T^* N} \theta) \in L|_N. \tag{4.9}$$

Accordingly, the pair defined by $(Z, \theta) \in L$ in $\iota^*(L)$ is the same as that defined by $(Z, \text{pr}_{T^* N} \theta)$ and may be identified with the latter. This justifies the equalities (4.8) and proves the proposition. \square

Remark 4.1. If $L = L(P)$, where P is a Poisson bivector field, condition (4.6) reduces to the almost-coupling condition of the Poisson case (see the beginning of Section 2 or [18]) and the manifold is a *Poisson–Dirac submanifold with a Dirac projection* in the sense of [5].

Furthermore, along any submanifold N of (M, L) we may define a field of subspaces $H_x(L, N)$, $x \in N$ by formula (2.10) with condition $\alpha \in \text{ann } F_x$ changed to $\alpha \in \text{ann } T_x N$. This leads to a notion that corresponds to coupling namely,

Definition 4.2. The submanifold N is a *cosymplectic submanifold* of the Dirac manifold (M, L) if, $\forall x \in N$, $T_x M = H_x(L, N) \oplus T_x N$.

The reason for this name is that if $L = L(P)$ for a Poisson bivector field P then N is a cosymplectic submanifold in the sense of [22]. Obviously, now, all the results stated in Proposition 2.2 are true along N . In particular, $L|_N$ has the expression (2.15) (with F^* replaced by $T^* N$) and the induced structure $L(N)$ is Poisson and defined by the bivector field Π of (2.15).

In what follows, we define a restriction of the Courant bracket of L to a submanifold $\iota : N \hookrightarrow (M, L)$. Like for Poisson–Dirac submanifolds [5], we may define

$$A_N(M, L) = \{(X, \alpha) \in L|_N / X \in TN\}, \tag{4.10}$$

which is important because, by (4.1), we have

$$L(N) = \iota^*(L) = \{(X, \iota^* \alpha) / (X, \alpha) \in A_N(M, L)\}. \tag{4.11}$$

Even though $A_N(M, L)$ may not be a vector bundle, we will consider the real, linear space $\Gamma A_N(M, L)$ of differentiable cross-sections of $A_N(M, L)$ (which may be zero). Using a partition of unity that consists of a tubular neighborhood of N and open sets that do not

intersect N , it follows easily that any cross-section $(X, \alpha) \in \Gamma A_N(M, L)$ admits extensions $(\tilde{X}, \tilde{\alpha}) \in \Gamma L$. Accordingly, on $\Gamma A_N(M, L)$ we may define a bracket

$$[(X, \alpha), (Y, \beta)]_A = ([(\tilde{X}, \tilde{\alpha}), (\tilde{Y}, \tilde{\beta})]_L)|_N. \tag{4.12}$$

Proposition 4.2. *The bracket (4.12) is well defined and, together with the projection $\rho(X, \alpha) = X$, yields a structure of Herz–Reinhart Lie algebra over $(\mathbb{R}, C^\infty(N))$ on $\Gamma A_N(M, L)$.*

Proof. In order to prove that the bracket (4.12) does not depend on the choice of the extensions it suffices to prove that it vanishes if, say, $(Y, \beta) = (0, 0)$ (see the proof of Theorem 2.1 of [11]). If

$$(\tilde{Y}, \tilde{\beta}) = \sum_{i=1}^n \tilde{\lambda}_i (\tilde{B}_i, \tilde{\theta}_i),$$

where $(\tilde{B}_i, \tilde{\theta}_i)$ is a local basis of L and $\tilde{\lambda}_i|_N = 0$, and since $X \in \mathcal{V}^1(N)$, we get

$$([\tilde{X}, \tilde{\alpha}), (\tilde{Y}, \tilde{\beta})]_L|_N = \sum_{i=1}^n (\tilde{\lambda}_i [(\tilde{X}, \tilde{\alpha}), (\tilde{B}_i, \tilde{\theta}_i)]_L)|_N + \sum_{i=1}^n (X \tilde{\lambda}_i) (\tilde{B}_i, \tilde{\theta}_i)|_N = 0.$$

The last assertion of the proposition is obvious if we recall that a Herz–Reinhart Lie (HRL) algebra (a pseudo-Lie algebra in the sense of [14]) \mathcal{R} over $(\mathbb{R}, C^\infty(N))$ is a real Lie algebra, which is a $C^\infty(N)$ -module endowed with a homomorphism $\rho : \mathcal{R} \rightarrow \mathcal{V}^1(N)$, such that the properties of the algebra of global cross-sections of a Lie algebroid hold. \square

Furthermore, the mapping $(X, \alpha) \mapsto (X, \iota^* \alpha)$ defines a HRL-algebra morphism

$$\iota^\# : \Gamma A_N(M, L) \rightarrow \Gamma(L(N)) \quad (L(N) = \iota^*(L)) \tag{4.13}$$

and $\ker \iota^\# = \Gamma(L \cap \text{ann } TN)$.

Proposition 4.3. *If $(N, \nu N)$ is a properly normalized submanifold of (M, L) , $A_N(M, L)$ is a differentiable field of subspaces of the fibers of $TN \oplus T_N^*M$.*

Proof. For any $(X, \alpha) \in A_N(M, L)$, we may write

$$(X, \alpha) = \sum_{i=1}^n \lambda_i (\text{pr}_{TN} B_i, \text{pr}_{T^*N} \theta_i) + (0, \text{pr}_{\nu^*N} \alpha), \tag{4.14}$$

where (B_i, θ_i) is a local basis of $L|_N$. Hence, the local cross-sections of $A_N(M, L)$ are spanned by differentiable cross-sections. \square

As a consequence of Proposition 4.3 and of formula (4.11), in the case of a properly normalized submanifold the morphism $\iota^\#$ is surjective, and we have the following exact sequence of HRL-algebras

$$0 \rightarrow \Gamma(L \cap \text{ann } TN) \rightarrow \Gamma A_N(M, L) \rightarrow \Gamma(L(N)) \rightarrow 0. \tag{4.15}$$

Along a properly normalized submanifold $(N, \nu N)$, all the vector fields and differential forms split into TN and νN -components, and we may identify ΓT^*N with the space of the tangent components of 1-forms in ΓT_N^*M . Accordingly, we may write a decomposition formula

$$[(X, \lambda), (Y, \mu)]_A = [(X, \lambda), (Y, \mu)]_{L(N)} + (0, B((X, \lambda), (Y, \mu))), \tag{4.16}$$

where $\lambda, \mu \in \text{pr}_{T^*N}(T_N^*M)$, $B((X, \lambda), (Y, \mu)) \in \nu^*N$ and $\forall Z \in \nu N$ one has

$$B((X, \lambda), (Y, \mu))(Z) = Z(\tilde{\lambda}(\tilde{Y})) - \lambda([\tilde{Z}, \tilde{Y}]) + \mu([\tilde{Z}, \tilde{X}]), \tag{4.17}$$

where \tilde{Z} is an extension of Z to M . The result does not depend on the choice of the extension \tilde{Z} because $B((X, \lambda), (Y, \mu))$ is a 1-form. By an analogy with Riemannian geometry to be explained below, we call B the *second fundamental form* of $(N, \nu N)$.

Let N be a Poisson–Dirac submanifold of the Poisson manifold (M, P) , which is properly normalized by the normal bundle νN and has the induced Poisson structure Π . Then, the kernel condition $K(N) = 0$ (see (4.4)) becomes

$$TN \cap \sharp_P(\text{ann } TN) = 0, \tag{4.18}$$

or, by passing to the annihilator spaces,

$$\text{ann } TN + A_N(M, P) = T_N^*M, \tag{4.19}$$

where

$$A_N(M, P) = \{\xi \in T_N^*M / \sharp_P \xi \in TN\} = \text{ann}(\sharp_P(\text{ann } TN)) \approx A_N(M, L(P)). \tag{4.20}$$

Furthermore, the bracket (4.12) produces a bracket of 1-forms

$$\{\alpha, \beta\}_A = \{\tilde{\alpha}, \tilde{\beta}\}_P|_N \quad (\alpha, \beta \in \Gamma T^*N), \tag{4.21}$$

where the P -bracket is that of the cotangent Lie algebroid of (M, P) and tilde denotes extension to M . Formula (4.16) becomes

$$\{\alpha, \beta\}_A = \{\alpha, \beta\}_\Pi + B(\alpha, \beta), \tag{4.22}$$

where the second fundamental form B is given by

$$B(\alpha, \beta)(Z) = -(L_{\tilde{Z}}P)|_N(\alpha, \beta), \quad Z = \tilde{Z}|_N \in \Gamma \nu N. \tag{4.23}$$

Now, the Riemannian terminology used above is justified as follows. A Riemannian metric g of the Poisson manifold (M, P) yields a canonical *Riemannian, contravariant derivative* D^P [1]. This is a cotangent-Lie algebroid-connection on TM which preserves the metric and has no torsion i.e.,

$$(\sharp_P \gamma)(g(\alpha, \beta)) = g(D_\gamma^P \alpha, \beta) + g(\alpha, D_\gamma^P \beta), \tag{4.24}$$

$$D_\alpha^P \beta - D_\beta^P \alpha = \{\alpha, \beta\}_P, \tag{4.25}$$

for all $\alpha, \beta, \gamma \in \Omega^1 M$. This operator is provided by the usual algebraic trick that derives the Riemannian connection from the metric, and the result is

$$\begin{aligned} 2g(D_\alpha^P \beta, \gamma) &= (\sharp_P \alpha)(g(\beta, \gamma)) + (\sharp_P \beta)(g(\gamma, \alpha)) - (\sharp_P \gamma)(g(\alpha, \beta)) + g(\{\alpha, \beta\}_P, \gamma) \\ &\quad + g(\{\gamma, \alpha\}_P, \beta) + g(\{\gamma, \beta\}_P, \alpha). \end{aligned} \tag{4.26}$$

Now, assume that $(N, \nu N = T^{\perp_g} N)$ is a properly normalized Poisson–Dirac submanifold with the induced Poisson structure Π . Then N has its own canonical operator D^Π on T^*N and, also, a contravariant derivative $D^{P,N}$ defined by

$$D_\alpha^{P,N} \beta = (D_{\tilde{\alpha}}^P \tilde{\beta})|_N, \quad \alpha, \beta \in \Gamma T^*N, \tag{4.27}$$

where $\tilde{\alpha}, \tilde{\beta}$ are extensions of α, β . It follows easily from (4.26) that the result of (4.27) does not depend on the choice of the extension. The formula

$$D_\alpha^{P,N} \beta = D_\alpha^\Pi \beta + \Psi(\alpha, \beta), \quad \Psi \in \Gamma \otimes^2 TM, \tag{4.28}$$

is a Gauss-type equation and Ψ is the g -second fundamental form of N . But, (4.24) shows that Ψ is determined by the form B of (4.22). Namely, we get

$$-2g(\Psi(\alpha, \beta), \gamma) = g(B(\alpha, \beta), \gamma) + g(B(\gamma, \alpha), \beta) + g(B(\gamma, \beta), \alpha). \tag{4.29}$$

The tensor field Ψ is not symmetric and its skew-symmetric part is $(1/2)B$.

Remark 4.2. For any normalized submanifold $(N, \nu N)$ of any manifold M , there exist Riemannian metrics g of M such that $\nu N = T^{\perp_g} N$. To get one, it suffices to define it along N , then extend to M along an open covering that consists of a tubular neighborhood of N and of sets that do not intersect N via a partition of unity. Then, if M is endowed with a Poisson structure P , it follows easily that $(N, \nu N)$ is a properly normalized Poisson–Dirac submanifold iff N is invariant by $\Phi = \sharp_P \circ \flat_g$. In particular, if M is a Kähler manifold, Φ is the complex structure tensor and the properly normalized Poisson–Dirac submanifolds are the complex analytic submanifolds of M .

Proposition 4.4. *A cosymplectic submanifold N of a Dirac manifold (M, L) has a vanishing second fundamental form.*

Proof. For a cosymplectic submanifold N , Corollary 2.1 holds for TN instead of F and, in particular, $L \cap \text{ann } TN = 0$. Then, by (4.9), $(X, \alpha) \in A_N(M, L)$ implies $(0, \text{pr}_{\nu^* N} \alpha) \in L$, hence, we must have $\text{pr}_{\nu^* N} \alpha = 0$. Therefore, $A_N(M, L) = L(N)$, $\iota^\#$ is an isomorphism, and $B = 0$. \square

Definition 4.3. A submanifold N of a Dirac manifold (M, L) which has a normal bundle νN such that $(N, \nu N)$ is properly normalized and has a vanishing second fundamental form will be called a *totally Dirac submanifold*.

In the Poisson case, these submanifolds were called Dirac [23] or Lie–Dirac [5]. We took the term *totally* from Riemannian geometry (totally geodesic submanifolds).

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